Functions

Limits (part 3)

How to find limit at a point by calculation?

❖ To find the limit at a point, substitute x by its value.

Example:

❶ $f(x) = 2x + 3 \qquad a = 3$

$$\lim_{x \to 3} f(x) = 2(3) + 3 = 6 + 3 = 9$$

How to find limit at a point by calculation?

❷ $f(x) = \frac{1}{x}$ $\quad a = 0$

$$\lim_{x\to 0} f(x) = \frac{1}{0} =$$

$$\lim_{x\to 0^-} f(x) = \frac{1}{0^-} = -\infty$$

$$\lim_{x\to 0^+} f(x) = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 0} f(x) = \nexists$$

⚠ $\frac{nb}{0} = \pm\infty$

How to find limit at a point by calculation?

❸ $f(x) = \frac{1}{x^2-1}$ $a = 1$

$$(1^-)^2 = 1^- \text{ so } 1^- - 1 = 0^-$$

$$\lim_{x \to 1^-} f(x) = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 1} f(x) = \frac{1}{1^2-1} = \frac{1}{0}$$

$$\lim_{x \to 1^+} f(x) = \frac{1}{0^+} = +\infty$$

$$\lim_{x \to 1} f(x) = \nexists$$

$$(1^+)^2 = 1^+ \text{ so } 1^+ - 1 = 0^+$$

⚠ $\frac{nb}{0} = \pm\infty$

How to find limit at a point by calculation?

❹ $f(x) = \frac{1}{x^2-1}$ $\quad a = -1$

$(-1^-)^2 = 1^+$ so $1^+ - 1 = 0^+$

$$\lim_{x\to -1^-} f(x) = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to -1} f(x) = \frac{1}{1^2-1} = \frac{1}{0}$$

$$\lim_{x\to -1^+} f(x) = \frac{1}{0^-} = -\infty$$

$(-1^+)^2 = 1^-$ so $1^- - 1 = 0^-$

$$\lim_{x\to 1} f(x) = \nexists$$

$\frac{nb}{0} = \pm\infty$

How to find limit at a point by calculation?

❺ $f(x) = \frac{1}{(x-2)^2}$ $\quad a = 2$

$$\lim_{x \to 2^-} f(x) = \frac{1}{(2^- - 2)^2} = \frac{1}{(0^-)^2} = \frac{1}{0^+} = +\infty$$

$$\lim_{x \to 2} f(x) = \frac{1}{(2-2)^2} =$$

$$\lim_{x \to 2} f(x) = +\infty$$

$$\lim_{x \to 2^+} f(x) = \frac{1}{(2^+ - 2)^2} = \frac{1}{(0^+)^2} = \frac{1}{0^+} = +\infty$$

⚠ $\frac{nb}{0} = \pm\infty$

How to find limit at a point by calculation?

❻ $f(x) = \frac{x^2+2x-3}{x+3}$ $\quad a = -3$

$$\lim_{x \to -3} f(x) = \frac{(-3)^2+2(-3)-3}{-3+3} = \frac{0}{0}$$ I.F.

- In this case we need to remove the common factor in the numerator and in the denominator.

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{(x-1)(x+3)}{x+3} = \lim_{x \to -3} x - 1 = -3 - 1 = -4$$

How to find limit at a point by calculation?

❼ $f(x) = \frac{-2x^2+3x-1}{x^2-1}$ $\qquad a = 1$

$$\lim_{x\to 1} f(x) = \frac{-2(1)^2+3(1)-1}{(1)^2-1} = \frac{0}{0}$$

I.F.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{-2(x-1)(x-\frac{1}{2})}{(x-1)(x+1)} = \lim_{x\to -3} \frac{-2\left(x-\frac{1}{2}\right)}{x+1} = \frac{-2\left(1-\frac{1}{2}\right)}{2} = -\frac{1}{2}$$

How to find limit at a point by calculation?

❽ $f(x) = \frac{x^3-8}{x^2-16}$ $\qquad a = 2$

$$\lim_{x\to 2} f(x) = \frac{(2)^3-8}{2^2-16} = \frac{0}{-12} = 0$$

How to find limit at a point by calculation?

❽ $f(x) = \frac{\sqrt{x+1}-1}{x}$ $\quad a = 0$

$\lim_{x \to 0} f(x) = \frac{0}{0}$ I.F.

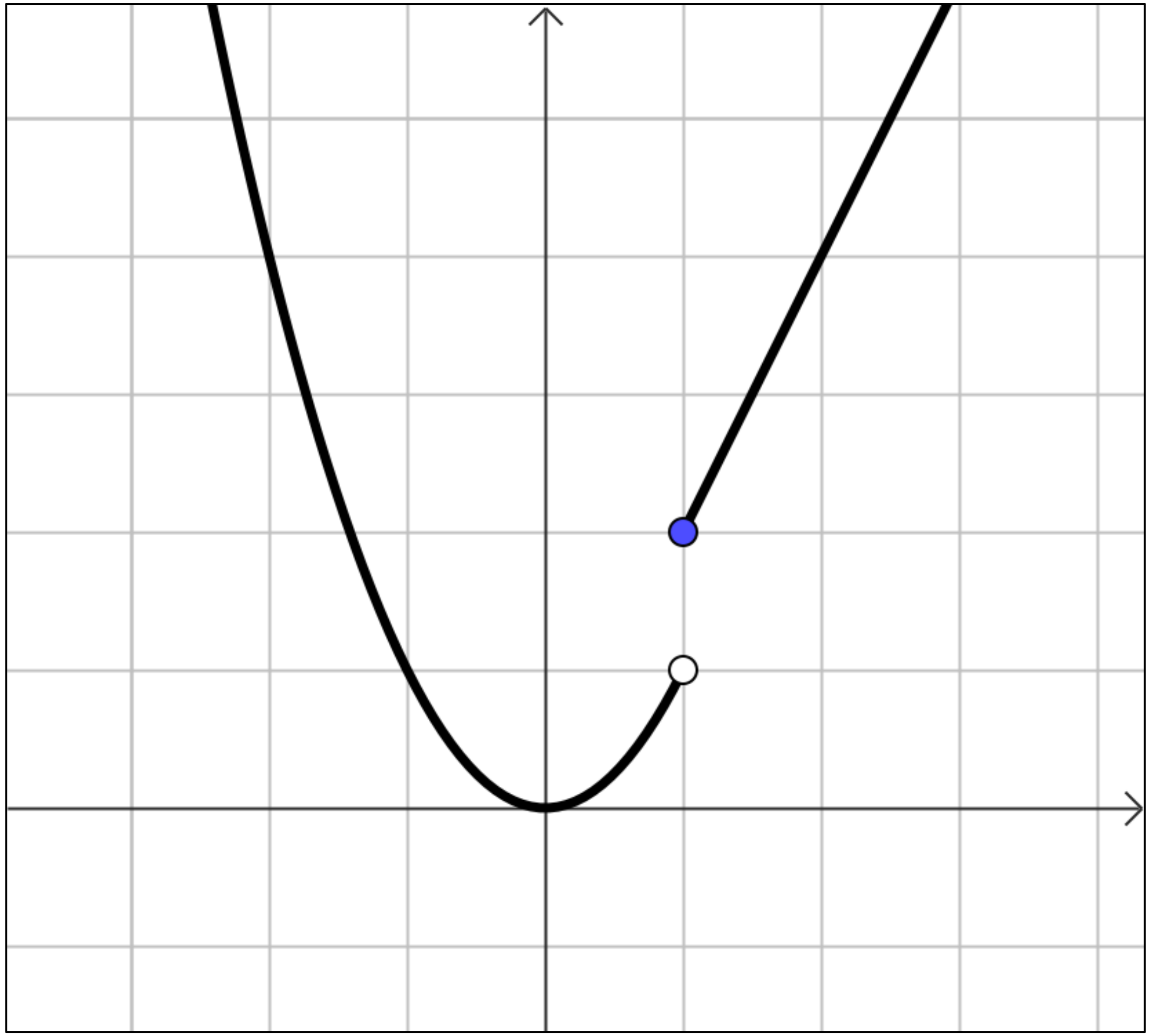
$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \lim_{x \to 0} \frac{x+1-1}{x(\sqrt{x+1}+1)} = \lim_{x \to 0} \frac{x}{x(\sqrt{x+1}+1)} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$

Continuity at a point

A function f is continuous at $x = a$ when the curve of f doesn't contain any form of discontinuity:

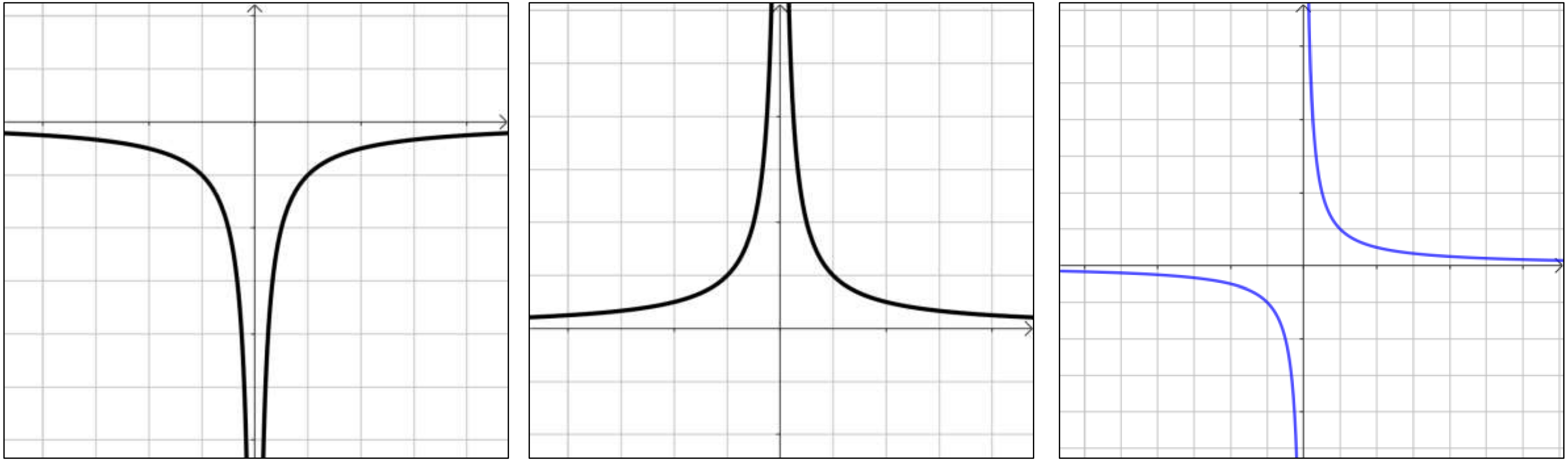
❶ Jump discontinuity



Continuity at a point

A function f is continuous at $x = a$ when the curve of f doesn't contain any form of discontinuity:

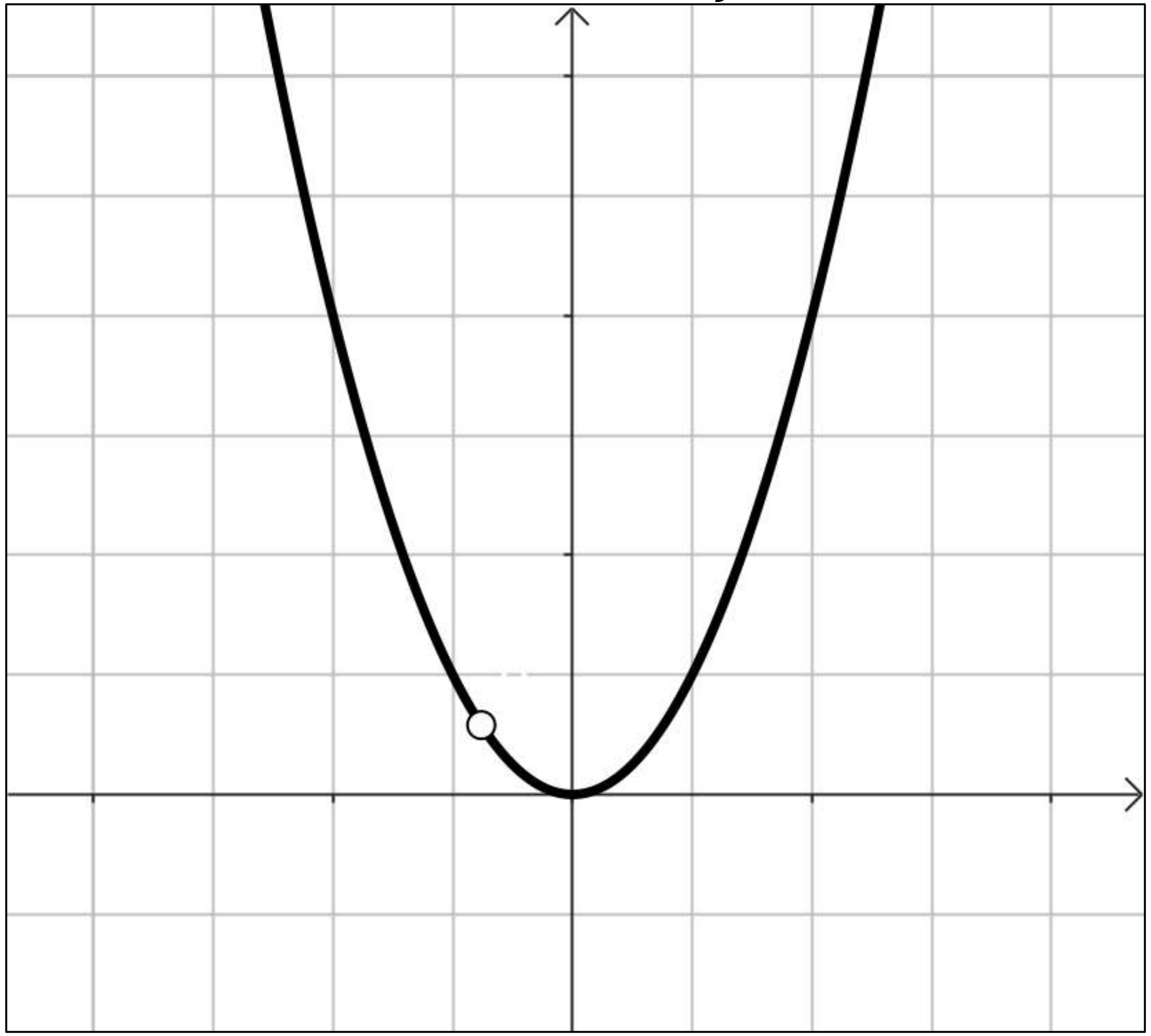
❷ Infinite discontinuity



Continuity at a point

A function f is continuous at $x = a$ when the curve of f doesn't contain any form of discontinuity:

❸ removable (gap) point



Continuity at a point

A function f is continuous at $x = a$ when:

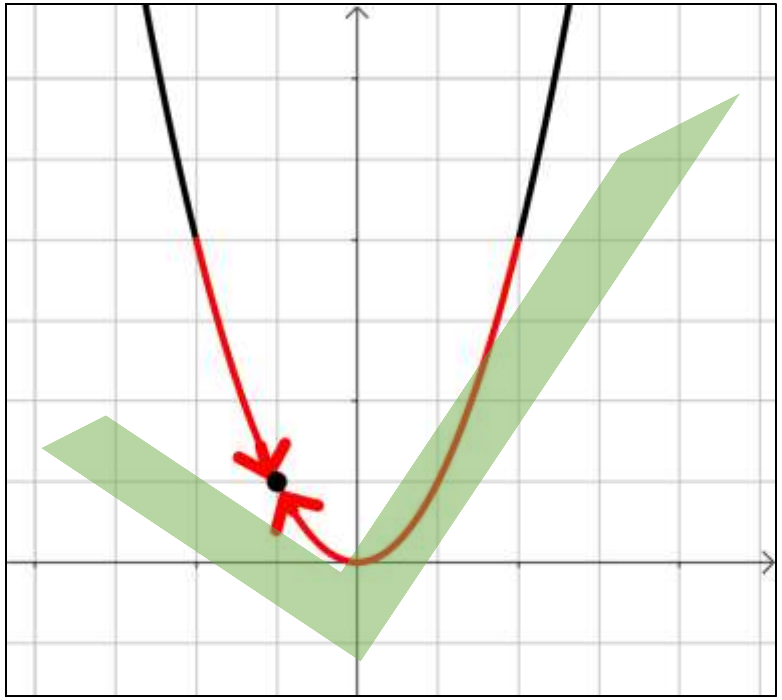
❶ f is defined at $x = a$ ($f(a)$ exists).

❷ $\lim_{x \to a} f(x)$ exist : $(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x))$

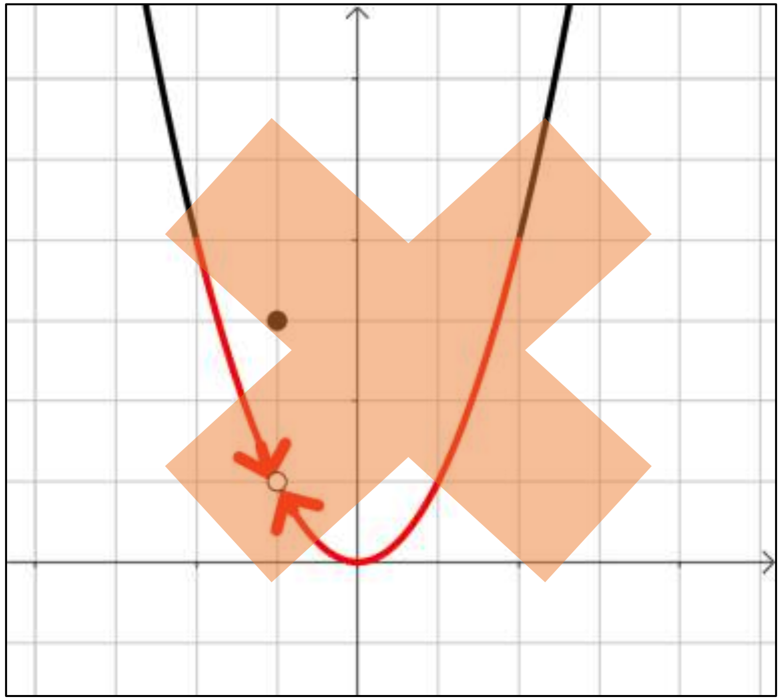
❸ $\lim_{x \to a} f(x) = f(a)$

$(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a))$

Continuity at a point

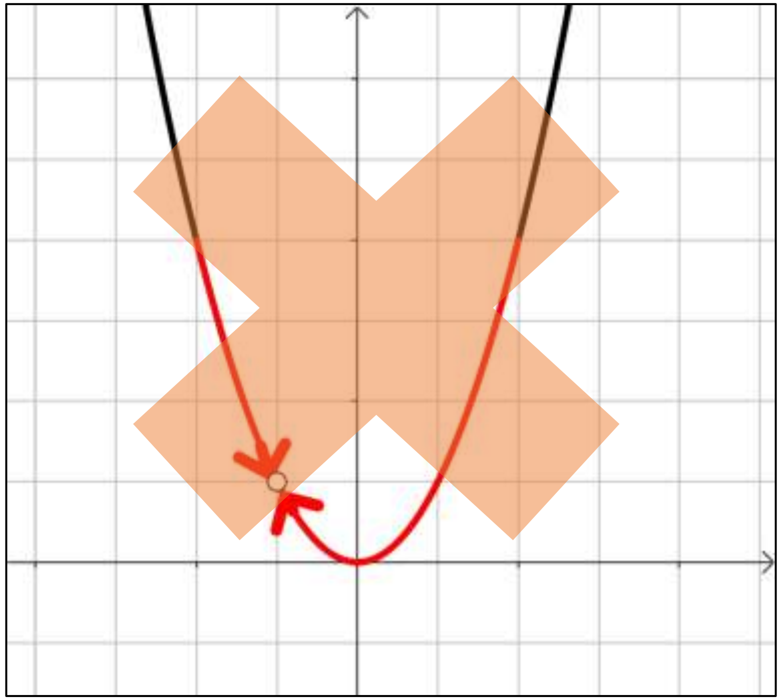


$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$



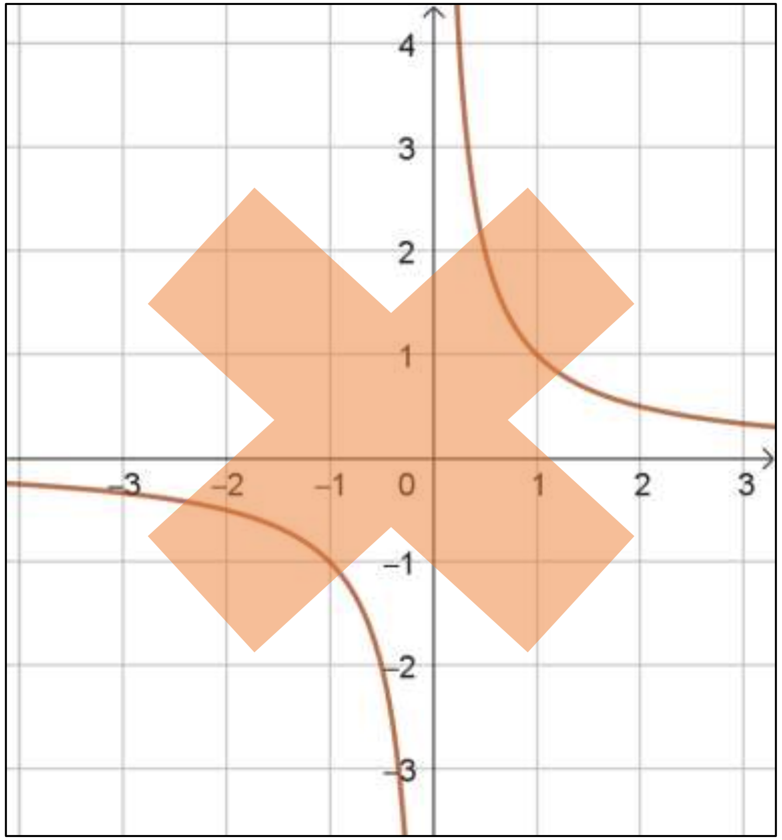
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$$

Continuity at a point



$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$f(a) \nexists$$

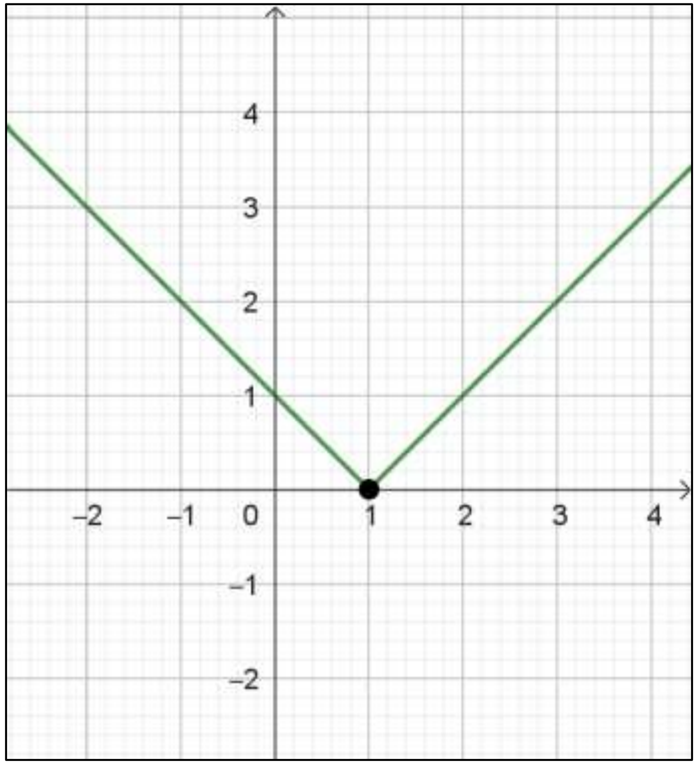


$$f(a) \nexists$$

Continuity at a point

Example.

❶ $f(x) = |x - 1| \quad ; \quad a = 1$



$f(1) = |1 - 1| = |0| = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - x = 1 - 1^- = 0^+$

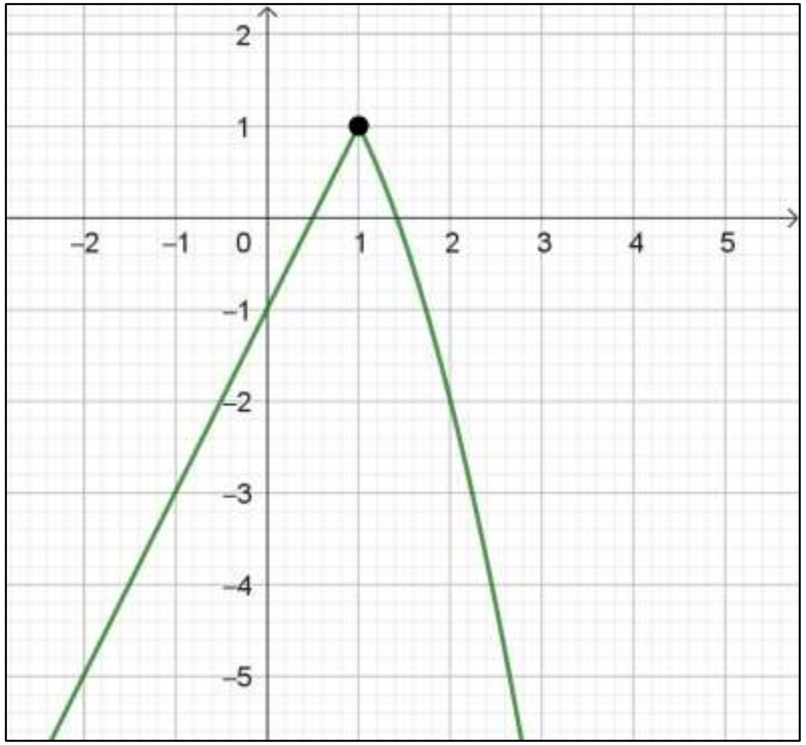
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x - 1 = 1^+ - 1 = 0^+$

So $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 0$ then f is continuous at x=1

Continuity at a point

Example ❷

$$f(x) = \begin{cases} -x^2 + 2 & if\ x > 1 \\ 2x - 1 & if\ x < 1 \\ 1 & if\ x = 1 \end{cases}$$



$f(1) = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x - 1) = 2 - 1 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-x^2 + 2) = -1 + 2 = 1$

f is continuous at $x = 1$

Continuity at a point

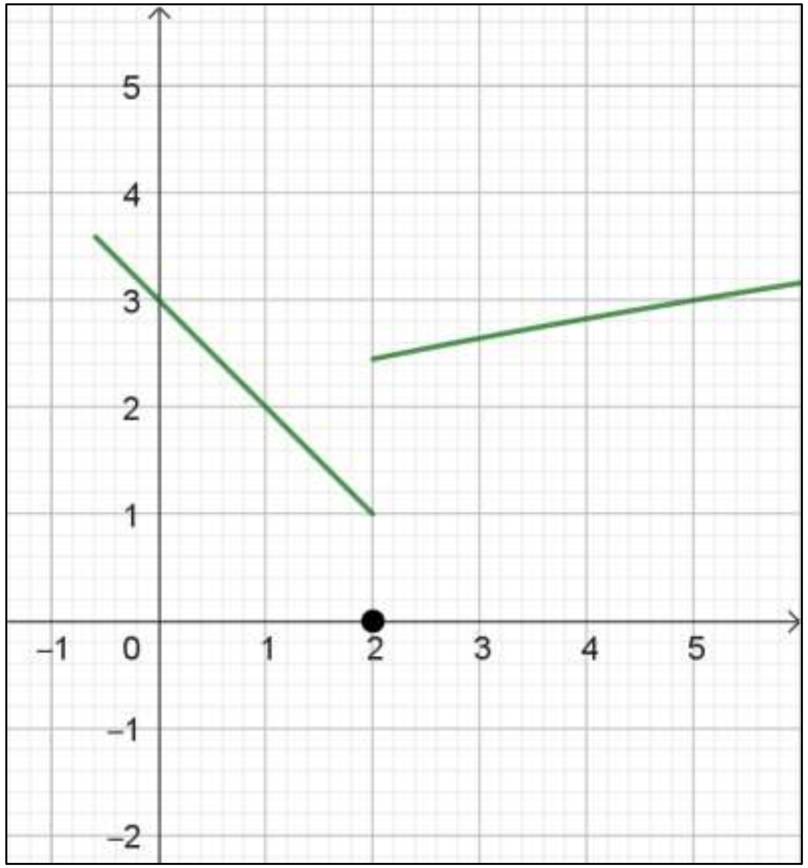
Example ❸

$$f(x) = \begin{cases} -x+3 & if\ x<2 \\ 0 & if\ x=2 \\ \sqrt{x+4} & if\ x>2 \end{cases}$$

$f(2) = 0$

$\underset{x\to 2^-}{\text{li}}\ f(x) = \lim_{x\to 2^-}(-x+3) = -2+3 = 1$

f is not continuous at $x = 2$

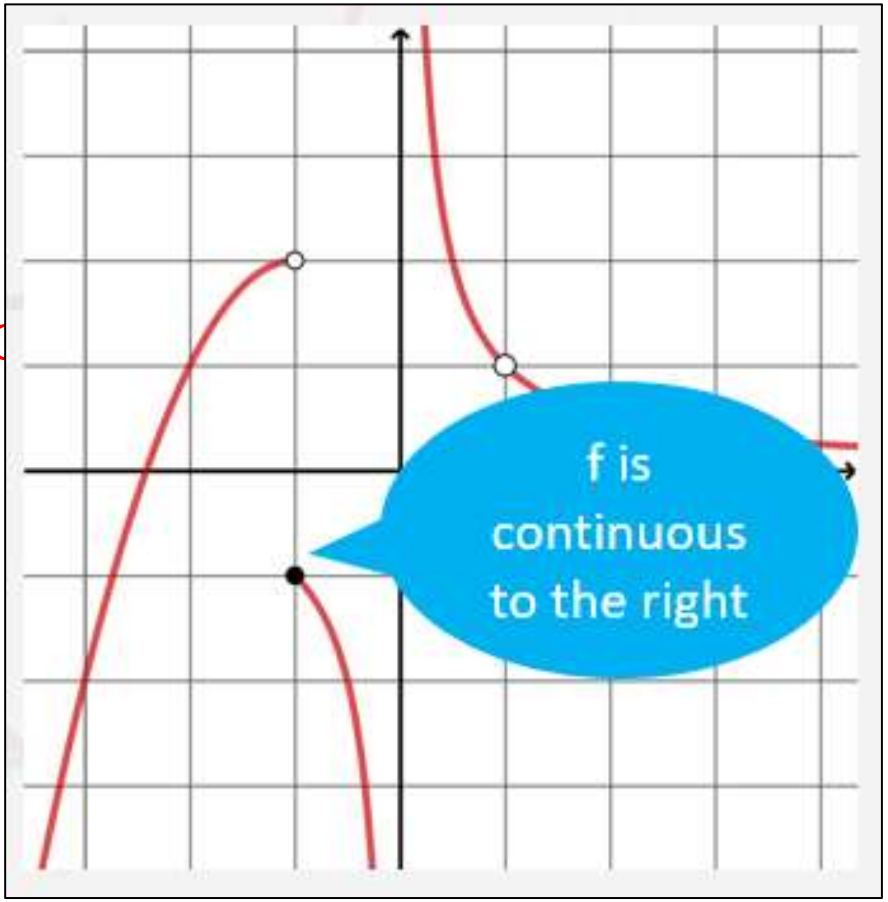


Continuity at a point

Remark:

If f is not continuous at $x = a$ and:

- if $\lim_{x \to a^-} f(x) = f(a)$, then f is continuous to the left (from below)
- if $\lim_{x \to a^+} f(x) = f(a)$, then f is continuous to the right (from above)



Time for practice

$$\lim_{x\to 5}\frac{x+5}{x+3}$$ $\frac{5}{4}$

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$$ -1

$$\lim_{x\to 2}\frac{-2x^2+1}{(x-2)^2}$$ $-\infty$

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}$$ 0

$$\lim_{x\to 3}\frac{x^2-3x}{x-3}$$ 3

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$ $\frac{1}{4}$